Electronic Notes in Theoretical Computer Science 1 (1995)
URL: http://www.elsevier.nl/locate/entcs/volume1.html 12 pages

Solutions of Functorial and Non-Functorial Metric Domain Equations [1]

F. Alessi [a] P. Baldan [a] G. Bellè [a] J. J. M. M. Rutten [b]

[a] *Dipartimento di Matematica ed Informatica, via Zanon 6, 33100 Udine, Italy*

[b] *CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*

Abstract

A new method for solving domain equations in categories of metric spaces is studied. The categories $\mathbf{CMS}^{\approx}$ and $\mathbf{KMS}^{\approx}$ are introduced, having complete and compact metric spaces as objects and *ϵ-adjoint pairs* as arrows. The existence and uniqueness of fixed points for certain endofunctors on these categories is established. The classes of complete and compact metric spaces are considered as pseudo-metric spaces, and it is shown how to solve domain equations in a non-categorical framework.

1 Introduction

Following [2,5], metric spaces have in the recent past often been used in the semantics of concurrent programming languages, as an alternative to the more standard partially ordered domains. One of the main problems that has been addressed, is to find solutions of recursive domain equations, that has become standard for ordered spaces since [13]. In all of the following papers, such solutions are constructed as fixed points of functors on (a subcategory of) a category of complete metric spaces: [8], [3], [10], [11], [4]. In spite of several (more and less important) differences, all these constructions are based on the same technique, which can be summarized as follows. As in the order-theoretic approach, *embedding-projection pairs* (ep-pairs) are used. The existence of such an ep-pair between two spaces is interpreted as an indication that the one approximates the other. Moreover, in the metric case a number can be assigned to such an ep-pair that actually measures the quality of this approximation. This measure is fundamental for the definition of Cauchy tower, which corresponds to that of *Cauchy sequence* in metric spaces. Next fixed points of functors that satisfy a property similar to contractiveness, are obtained as colimits of such Cauchy towers.

One can consider all of the above results as a 'kind of categorical generalization' of Banach-Caccioppoli's fixed point theorem for contractions on

[1] Work partially supported by ECC Science project MASK and by MURST 40% grants.

complete metric spaces. Although certain notions of metric spaces, such as Cauchy sequence, have been generalized to categories of metric spaces by means of ep-pairs, these categories have never been considered as *large metric spaces* themselves in such a way, that Banach's theorem could be directly applied This is what we intend to do in the present paper. (A similar programme has been carried out in the context of *hyperuniverses*, see [7]).

It is well known for ordered spaces that the more general *adjoint* pairs can be used as an alternative to ep-pairs. Motivated by Lawvere's enriched-categorical view on metric spaces [9], the notion of ϵ*-adjoint* pair has recently been introduced in [12]. At the same time, a new approach to the construction of fixed points of functors was introduced in [1], based on the use of ϵ*-isometries* rather than ep-pairs. Interestingly, the notions of ϵ-adjoint pair and ϵ-isometry coincide (Section 2), and they will serve here as the basis for a construction of fixed points of functors in a way that mimics the order-theoretic approach. More specifically, the categories $\mathbf{CMS}^{\approx}$ and $\mathbf{KMS}^{\approx}$ are introduced (in Section 3), having complete and compact metric spaces as objects, and ϵ-adjoint pairs as arrows. By extending the definitions and results of [3], fixed point theorems for contractive functors on these categories are obtained. Although in this way no new solutions are found compared to the previous approaches with ep-pairs (as in the order-theoretic case), the use of adjoint pairs leads to some further observations, in Section 4, which contains the main contribution of the present paper.

We shall see how ϵ-adjoint pairs can be used to turn the class of complete metric space into a *large pseudo-metric* space. (In a pseudo-metric space, two different elements can have distance 0.) For (small or large) pseudo-metric spaces that are complete (in the usual sense), a weak version of Banach's theorem holds: for every *contraction* f there exists an element x such that the distance between x and $f(x)$ is 0. It can be shown that the pseudo-metric space of all complete metric spaces, is complete, again in the usual (metric, not categorical) sense. By applying Banach's theorem to a (large) contractive function T on this pseudo-metric space, we find the existence of a complete metric space X such that the distance between X and TX equals zero. (Note that here contractive is used in the standard sense, not in some categorical variant of it. Also note that T need not be functorial.) For *compact* metric spaces one can prove that having distance 0 implies being isomorphic. As a consequence, any (large) contraction on the class of all compact metric spaces has a fixed point that is unique up to isomorphism. (The categorical results on the existence and uniqueness of fixed points for contracting endofunctors on $\mathbf{KMS}^{\approx}$ can now be obtained as a corollary of this theorem.)

2 Metric adjoints pairs and isometries

In this section, we define the basic notions of ϵ-adjoint pair and ϵ-isometry. They have been introduced independently in [12] and [1], respectively, and are shown to be equivalent.

A 1-*bounded metric space* is a set X together with a mapping

$$X(-,-) : X \times X \to [0,1],$$

called a *metric*, which satisfies, for all x, y, and z in X:

(i) $X(x,x) = 0$,

(ii) $X(x,z) \leq X(x,y) + X(y,z)$,

(iii) $X(x,y) = X(y,x)$,

(iv) if $X(x,y) = 0$ then $x = y$.

If the mapping $X(-,-)$ satisfies only conditions 1, 2, and 3, then X is called a *pseudo-metric space*.

Let X, Y be metric spaces. A mapping $f : X \to Y$ is *non-expansive* if for all x and x' in X, $X(x,x') \geq Y(f(x), f(x'))$. The set of non-expansive mappings between two metric spaces X and Y, $Y^X = \{f : X \to Y \mid f \text{ is non-expansive } \}$, can be supplied with a metric

$$Y^X(f,g) = \sup\{Y(f(x), g(x)) \mid x \in X\}.$$

A mapping $f \in Y^X$ is *contracting* if there exists $0 \leq \epsilon < 1$ such that for all $x, x' \in X$, $\epsilon \cdot X(x,x') \geq Y(f(x), f(x'))$. A mapping $f \in Y^X$ is called an *isometric embedding* if $\forall x, x' \in X.\ X(x,x') = Y(f(x), f(x'))$. If f is also a bijection then it is an *isometry.*

The following notation will be used: for real numbers ϵ, r, and r' in $[0,1]$, let

$$r \approx_\epsilon r' \text{ if and only if } |r - r'| \leq \epsilon.$$

(We say that r and r' are equal 'modulo', or 'up to' ϵ.)

Definition 2.1 Let X and Y be metric spaces. Two non-expansive mappings $f : X \to Y$ and $g : Y \to X$ are ϵ-*adjoint*, denoted by $f \dashv_\epsilon g$, if for all x in X and y in Y,

$$Y(f(x), y) \approx_\epsilon X(x, g(y)).$$

If $f \dashv_0 g$ then $\langle f, g\rangle$ is called a *proper* adjoint pair, and we shall simply write $f \dashv g$.

Interestingly, ϵ-adjoint pairs turn out to be 'isometries up to ϵ', which are introduced next. Consider again a pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$, and define

$$\delta\langle f, g\rangle = \max\{X^X(1_X, g \circ f),\ Y^Y(f \circ g, 1_Y)\}.$$

Intuitively, this number gives a measure for how similar X and Y are.

Definition 2.2 Let X and Y be metric spaces. A pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ with $\delta\langle f, g\rangle \approx_\epsilon 0$ is called an ϵ-*isometry.*

Note that by definition, any pair $\langle f, g\rangle$ of non-expansive mappings is an ϵ-isometry, for $\epsilon = \delta\langle f, g\rangle$.

The above definition can be justified by the observation that 0-isometries satisfy, due to axiom (iv) in the definition of a metric, $1_X = g \circ f$ and $f \circ g = 1_Y$, and consequently f (and also g) is an isometry.

The following theorem states the equivalence of the notions of ϵ-adjoint and ϵ-isometry. In particular proper adjoint pairs are isometries.

Theorem 2.3 *Let X and Y be metric spaces, and ϵ in $[0,1]$. For all non-expansive mappings $f : X \to Y$ and $g : Y \to X$,*

$$f \dashv_\epsilon g \quad \textit{iff} \quad \langle f, g\rangle \textit{ is an } \epsilon\textit{-isometry.}$$

Proof. Suppose $f \dashv_\epsilon g$. For any x in X, $X(x, g \circ f(x)) \approx_\epsilon Y(f(x), f(x)) = 0$, thus $X^X(1_X, g \circ f) \approx_\epsilon 0$. Similarly $Y^Y(f \circ g, 1_Y) \approx_\epsilon 0$. It follows that $\delta\langle f, g\rangle \approx_\epsilon 0$. Conversely, suppose $\delta\langle f, g\rangle \approx_\epsilon 0$. For all x in X and y in Y,

$$\begin{aligned} X(x, g(y)) &\leq X(x, g \circ f(x)) + X(g \circ f(x), g(y)) \\ &\leq \epsilon + X(g \circ f(x), g(y)) \\ &\leq \epsilon + Y(f(x), y). \end{aligned}$$

Similarly, $Y(f(x), y) \leq \epsilon + X(x, g(y))$. Thus $Y(f(x), y) \approx_\epsilon X(x, g(y))$. □

A consequence of this theorem is that any pair $\langle f, g\rangle$ of non-expansive mappings is an ϵ-adjoint pair, for $\epsilon = \delta\langle f, g\rangle$.

The proof in the theorem above uses only axioms 1 and 2 of the definition of a metric. Therefore the theorem also holds for what could be called *generalized* metric spaces, satisfying only axioms 1 and 2. This sheds at the same time some light on the relationship between the above and the standard notion of adjoints of (pre)ordered spaces. A preorder $(P, \leq_P)$ consists of a set P and a reflexive and transitive relation $\leq_P$ on P. Two monotone functions $f : P \to Q$ and $g : Q \to P$ are adjoint if for all x in P and y in Q: $f(x) \leq_Q y$ if and only if $x \leq_P g(y)$. This definition is a special instance of the metric notion of (proper) adjoint pair, because any preorder $(P, \leq_P)$ induces a generalized metric space as follows: for x and x' in P,

$$P(x, x') = \begin{cases} 0 \text{ if } x \leq_P x' \\ 1 \text{ if } x \not\leq_P x'. \end{cases}$$

(Reflexivity and transitivity of $\leq_P$ imply axioms 1 and 2, and every monotone mapping is non-expansive with respect to the induced metrics.) The above theorem gives for preorders P and Q the well-known alternative definition of adjoint pair: $1_P \leq_P g \circ f$ and $f \circ g \leq_Q 1_Q$.

Although the notions of ϵ-adjoints and ϵ-isometries may be new, their definitions and in fact all of the above considerations are an immediate consequence of Lawvere's $\mathcal{V}$-categorical theory of metric spaces [9]. There metric spaces X are viewed as generalized categories, of which the hom functor $X(-,-)$ does not take values in the category of sets but in the (category of) real numbers. Recently, some of Lawvere's ideas have been further pursued with the aim of unifying traditional order-theoretic and metric domain theory, in [14], [6], and [12]. The latter paper deals, more specifically, with the aforementioned generalized metric spaces.

As was mentioned in the introduction, in the literature several techniques have been proposed for solving metric domain equations. All of them use *embedding-projection* pairs: pairs of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$. This definition is derived from that of embedding-projection pair between cpo's, as used in [13][2]. For an ep-pair $\langle f, g\rangle$, we have that $\delta\langle f, g\rangle = Y^Y(f \circ g, 1_Y)$, because $X^X(1_X, g \circ f) = 0$. As we saw above, any pair of non-expansive mappings is an ϵ-adjoint pair so ep-pairs are trivially ϵ-adjoint pairs. Note that between two spaces X and Y, an ϵ-adjoint pair always exists: just take any pair of non-expansive mappings between them. This is not the case for ep-pairs.

The use of ϵ-adjoint pairs rather than ep-pairs is crucial for the results of Section 4.

3 Categorical fixed point results

In this section, we present two categories of metric spaces whose morphisms are ϵ-adjoint pairs, and extend the results of [3]. First we introduce the concepts of Cauchy tower and contracting functor. Then we see that a contracting functor gives rise to a Cauchy tower and that the limit of this tower is a fixed point of the functor. Finally two results on uniqueness are presented, the first for functors on the category of complete metric spaces which are contracting and hom-contracting, and the second for functors on the category of compact metric spaces which are contracting.

Before introducing the categories we shortly recall some metric notions. Let X be a metric space. A sequence $(x_n)_n$ in X is a *Cauchy sequence* if for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$ we have $X(x_n, x_m) \leq \epsilon$. The metric space X is called *complete* whenever each Cauchy sequence converges to an element of X. It is called *compact* if each sequence contains a converging subsequence.

Definition 3.1 Let $\mathbf{CMS}^{\approx}$ denote the category that has non-empty complete metric spaces as objects and ϵ-adjoint pairs as arrows. The composition of a pair of arrows $\iota_1 = \langle i_1, j_1\rangle : X \to Y$ and $\iota_2 = \langle i_2, j_2\rangle : Y \to Z$ is defined as $\iota_2 \circ \iota_1 = \langle i_2 \circ i_1, j_1 \circ j_2\rangle : X \to Z$.

With $\mathbf{KMS}^{\approx}$ we denote the full subcategory of $\mathbf{CMS}^{\approx}$ that has compact metric spaces as objects.

Definition 3.2 (i) A *tower* in $\mathbf{CMS}^{\approx}$ is a sequence $(X_n, \iota_n)_n$ of objects and arrows such that for all n we have $X_n \stackrel{\iota_n}{\to} X_{n+1}$.

(ii) A tower $(X_n, \iota_n)_n$ in $\mathbf{CMS}^{\approx}$, with $\iota_n = \langle i_n, j_n\rangle$, is called a *Cauchy tower* if

$\forall \epsilon > 0. \exists n_0 \in \mathbb{N}.\ \forall m > n \geq n_0.\ \delta(\iota_{nm}) < \epsilon$,

where $\iota_{nm} = \iota_{m-1} \circ \ldots \circ \iota_n$.

(The following notation will be used below. For natural numbers $m > n$, $i_{nm} = i_{m-1} \circ \ldots \circ i_n$ and $j_{nm} = j_n \circ \ldots \circ j_{m-1}$.)

[2] Let D, E be cpo's. An embedding-projection pair consists of continuous functions $i : D \to E$ and $j : E \to D$ such that $i \circ j \sqsubseteq 1_E$ and $j \circ i = 1_D$.

Intuitively a Cauchy tower is a sequence of spaces such that for increasing n and $m > n$, X_n and X_m are more and more alike.

The direct limit construction of a Cauchy tower is defined as usual:

Definition 3.3 (Direct Limit Construction) Let $(X_n, \iota_n)_n$ be a Cauchy tower in $\mathbf{CMS}^{\approx}$, where $\iota_n = \langle i_n, j_n \rangle$. The *direct limit* of $(X_n, \iota_n)_n$ is a cone $(X, (\gamma_n)_n)$, where $\gamma_n = \langle \alpha_n, \beta_n \rangle$, which is defined as follows:

(a) The space X is given by
$X = \{(x_n)_n : \forall n \in \mathbb{N}.\ x_n \in X_n \text{ and } x_n = j_n(x_{n+1})\}$
equipped with a metric $X(-,-)$, such that for all $(x_n)_n, (x'_n)_n \in X$
$X((x_n)_n, (x'_n)_n) = \sup_n X_n(x_n, x'_n)$.

(b) Arrows γ_n are defined as
- $\alpha_n : X_n \to X$
  $\alpha_n(x) = (x_k)_k \quad \text{with} \quad x_k = \lim_{h\to\infty} j_{kh} \circ i_{nh}(x)$
- $\beta_n : X \to X_n$
  $\beta_n((x_k)_k) = x_n$.

Note that α_n is well defined, because $(j_{kh} \circ i_{nh}(x))_{h>\max\{k,n\}}$ is a Cauchy sequence. This follows from the fact that the tower $(X_n, \iota_n)_n$ is Cauchy. Moreover it is easy to show that X is a complete metric space and $(X, (\gamma_n)_n)$ is a cone for the tower $(X_n, \iota_n)_n$.

The following result gives a criterion for checking the initiality of a cone.

Lemma 3.4 (Initiality Lemma) *Let $(X_n, \iota_n)_n$ be a Cauchy tower in $\mathbf{CMS}^{\approx}$ and let $(X, (\gamma_n)_n)$, with $\gamma_n = \langle \alpha_n, \beta_n \rangle$, be a cone for that tower. Then*

$(X, (\gamma_n)_n)$ *is an initial cone* *iff* $\lim_{n\to\infty} \delta(\gamma_n) = 0$.

Proof. We shall first prove the implication from right to left. A corollary of this will be that the direct limit of a Cauchy tower, described above, is an initial cone. From that fact, the implication from left to right can be immediately derived.

Proof of $\Leftarrow$. Let $(X', (\gamma'_n)_n)$, with $\gamma'_n = \langle \alpha'_n, \beta'_n \rangle$ be another cone for $(X_n, \iota_n)_n$. We have to prove the existence of a unique arrow $\iota : X \to X'$ such that for all $n \in \mathbb{N}$, $\gamma'_n = \iota \circ \gamma_n$. The sequences $(\alpha'_n \circ \beta_n)_n$ and $(\alpha_n \circ \beta'_n)_n$ are Cauchy because $(X_n, \iota_n)_n$ is. Since X and X' are complete, we can define $i = \lim \alpha'_n \circ \beta_n$ and $j = \lim \alpha_n \circ \beta'_n$. This defines an arrow $\iota = \langle i, j \rangle$ from X to X'. It follows from the assumption that $\lim_{n\to\infty} \delta(\gamma_n) = 0$ that $\gamma'_n = \iota \circ \gamma_n$ and that ι is the unique arrow with this property. This proves that $(X, (\gamma_n)_n)$ is initial.

As a consequence, we have the following.

Corollary 3.5 *The direct limit of a Cauchy tower is an initial cone for that tower.*

This follows from the fact that $\lim_{n\to\infty} \delta(\gamma_n) = 0$ (with $(\gamma_n)_n$ as in Definition 3.3).

This corollary, at its turn, now yields an easy proof of the other implication of Lemma 3.4:

Proof of $\Rightarrow$. Let $(X', (\gamma'_n)_n)$, with $\gamma'_n = \langle \alpha'_n, \beta'_n \rangle$, be an initial cone for the Cauchy tower $(X_n, \iota_n)_n$. By the corollary above, the direct limit $(X, (\gamma_n)_n)$ is initial as well, thus X and X' are isomorphic. Therefore $(\delta(\gamma'_n))_n$ is equal to $(\delta(\gamma_n))_n$, which has limit 0. □

3.1 Fixed point theorems

Following [3], we introduce the notion of *contracting functor*, generalizing the notion of contracting function.

Definition 3.6 A functor $F : \mathbf{CMS}^{\approx} \to \mathbf{CMS}^{\approx}$ is called *contracting* if $\exists 0 \leq \epsilon < 1$ such that for every morphism $X_1 \xrightarrow{\iota} X_2$,

$$\delta(F\iota) \leq \epsilon \cdot \delta(\iota).$$

By the Initiality Lemma, a contracting functor preserves Cauchy towers and their initial cones, in a similar way as contracting functions preserve Cauchy sequences and their limits:

Lemma 3.7 *Let $F : \mathbf{CMS}^{\approx} \to \mathbf{CMS}^{\approx}$ be a contracting functor and $(X_n, \iota_n)_n$ a Cauchy tower with an initial cone $(X, (\gamma_n)_n)$. Then $(FX_n, F\iota_n)_n$ is a Cauchy tower with $(FX, (F\gamma_n)_n)$ as an initial cone.*

Using a standard categorical fixed point theorem (see e.g. [13] or [3], Theorem 3.14) we can prove the existence of fixed points for contracting functors on the category $\mathbf{CMS}^{\approx}$.

Theorem 3.8 (Existence of fixed point) *Let $F : \mathbf{CMS}^{\approx} \to \mathbf{CMS}^{\approx}$ be a contracting functor. Then F has a fixed point, that is, there exists a complete metric space X such that $X \cong FX$.*

Proof. Let X_0 be any complete metric space and let $X_0 \xrightarrow{\iota_0} FX_0$ be any arrow[3]. Consider the tower $(F^n X_0, F^n \iota_0)_n$. Because F is contracting, this is a Cauchy tower. Thus it has a direct limit $(X, (\gamma_n)_n)$, which is an initial cone for the tower. Moreover F preserves the tower and its initial cone. By the categorical fixed point theorem, $FX \cong X$. □

Remark 3.9 Notice that contractiveness is not a necessary condition in order that a functor has fixed points. For example the identity functor is not contracting.

As in [3], fixed points are unique (up to isomorphism) if the functor satisfies an additional condition of contractiveness on arrows.

Definition 3.10 A functor $F : \mathbf{CMS}^{\approx} \to \mathbf{CMS}^{\approx}$ is called *hom-contracting* whenever for all X_1, X_2 in $\mathbf{CMS}^{\approx}$ there exists $0 \leq \epsilon < 1$ such that

$$F_{|Hom(X_1,X_2)} : Hom(X_1, X_2) \to Hom(FX_1, FX_2)$$

is contracting with factor ϵ, where the Hom sets are supplied with the obvious metric.

[3] Such an arrow always exists, e.g. choose a pair of constant functions.

Theorem 3.11 (Existence and uniqueness of fixed point)
Let $F : \mathbf{CMS}^{\approx} \to \mathbf{CMS}^{\approx}$ be a contracting and hom-contracting functor. Then F has a unique fixed point up to isomorphism, that is there exists a complete metric space X such that

$$X \cong FX \text{ and } \forall X' \in \mathbf{CMS}^{\approx}.\ FX' \cong X' \Rightarrow X \cong X'.$$

Proof. As in [3]. □

All the results above for $\mathbf{CMS}^{\approx}$ also hold for $\mathbf{KMS}^{\approx}$. Moreover for compact metric spaces the sole contractiveness hypothesis is enough to ensure uniqueness of fixed points. This is an immediate consequence of Theorem 4.9 regarding the non-functorial case, which will be proved in the next section.

Theorem 3.12 (Existence and uniqueness of fixed point)
Let $F : \mathbf{KMS}^{\approx} \to \mathbf{KMS}^{\approx}$ be a contracting functor. Then F has a unique fixed point up to isomorphism. □

Remark 3.13 The usual domain constructors, such as shrinking ($\epsilon \cdot -$), product ($\times$), disjoint union ($\dot{\cup}$), function space ($\to$) and powerdomain ($\mathcal{P}_{cl}$) turn out to be functors in $\mathbf{CMS}^{\approx}$ ($\mathbf{KMS}^{\approx}$). As in [3], it is possible to associate to these functors a *contraction coefficient*, which allows to identify a large class of contracting (hom-contracting) functors that induce equations with a unique solution.

Remark 3.14 In [12], a category of generalized metric spaces is studied, having complete quasi (i.e., non-symmetric) ultrametric spaces as objects and ϵ-adjoint pairs as arrows. It contains both the category of complete partial orders and complete ultrametric spaces as full subcategories. Fixed point theorems for endofunctors on this category are given which generalize the standard order-theoretic and the metric approach.

4 A Pseudo-metric on Metric Spaces

In this section we leave the categorical framework. We use ϵ-adjoint pairs for defining a pseudo-metric Δ on the *class* $\mathcal{C}$ of complete metric spaces. An interesting fact is that $\mathcal{C}$ is complete w.r.t Δ, in the usual sense that every Cauchy sequence of complete metric spaces converges to a complete metric space. Thus the following "weak" version of Banach-Caccioppoli's theorem—which holds for (arbitrary) complete pseudo-metric spaces and can be proved along the same lines as Banach's theorem for complete *metric* spaces—can be applied to $\mathcal{C}$.

Theorem 4.1 *Let $f : X \to X$ be a contractive function on a complete pseudo-metric space X. Then there exists x in X such that $X(x, f(x)) = 0$. Moreover if $X(y, f(y)) = 0$, for $y \in X$, then $X(x, y) = 0$.* □

An application of this theorem will yield for every (large) contraction on $\mathcal{C}$ a complete metric space X with $\Delta(X, T(X)) = 0$. For the subclass $\mathcal{K}$ of $\mathcal{C}$ of compact metric spaces, the latter equality will be shown to imply $X \cong T(X)$.

We thus shall find unique fixed points (up to isomorphism) for contractions on the class $\mathcal{K}$ by an application of Banach's fixed point theorem.

Definition 4.2 For complete metric spaces X and Y, a *distance* is defined as follows:

$$\Delta(X,Y) = \inf\{\epsilon \mid \langle i,j\rangle : X \to Y \quad \text{is an } \epsilon\text{-adjoint pair}\}.$$

(Recall that any pair $\langle i,j\rangle$ of non-expansive mappings $i : X \to Y$ and $j : Y \to X$ is an ϵ-adjoint pair for $\epsilon = \delta\langle i,j\rangle$).

Lemma 4.3 *The class $\mathcal{C}$ with distance Δ is a (large) pseudo-metric space, i.e. for all complete metric spaces X, Y, and Z,*

(i) $\Delta(X,X) = 0$,

(ii) $\Delta(X,Z) \leq \Delta(X,Y) + \Delta(Y,Z)$,

(iii) $\Delta(X,Y) = \Delta(Y,X)$. □

Note that the use of ϵ-adjoint pairs is crucial for this lemma. In particular, (iii) would not hold if ep-pairs were used. Still (i) and (ii) would be valid, leaving us with a *generalized* metric (cf. [12]).

Remark 4.4 It is not true that $\Delta(X,Y) = 0$ implies $X \cong Y$: consider metric spaces $\tilde{X}$ and $\tilde{Y}$ defined as follows:

$$\tilde{X} = \{(n,i),(m,j) \mid n,m \geq 1,\ i,j \in \{1,2\}\};$$

$$\tilde{X}((n,i),(m,j)) = \begin{cases} 0 & \text{if } n = m,\ i = j; \\ 1 & \text{if } n \neq m; \\ 1/2^n & \text{if } n = m,\ i \neq j, \end{cases}$$

and $\tilde{Y} = \tilde{X} \cup \{p_0\}$. The distance on $\tilde{Y}$ is as for $\tilde{X}$, extended with $\tilde{Y}(p_0,(n,i)) = 1$ for each $(n,i) \in Y$. It is not difficult to prove that for each $\epsilon \geq 0$ there exists a ϵ-adjoint pair between $\tilde{X}$ and $\tilde{Y}$. Therefore $\Delta(\tilde{X},\tilde{Y}) = 0$. However $\tilde{X}$ is not isometric to $\tilde{Y}$.

The class $\mathcal{C}$ with pseudo-metric Δ is complete in the usual sense:

Proposition 4.5 *Let $(X_n)_n$ be a Cauchy sequence of complete metric spaces. Then there exists a complete metric space X such that*

$$\forall\epsilon \geq 0.\exists n_0 \in \mathbb{N}.\forall n \geq n_0.\Delta(X,X_n) \leq \epsilon$$

Proof. Consider a subsequence of $(X_n)_n$, say $(X'_k)_k$, such that $\Delta(X'_k, X'_{k+1}) \leq 1/2^k$. Choose for every k a $(1/2^{k-1})$-adjoint pair $\iota_k = \langle i_k, j_k\rangle : X'_k \to X'_{k+1}$ (which exists by the definition of Δ) and consider the tower $(X'_k, \iota_k)_k$. It is a Cauchy tower in the sense of Definition 3.2, so we can consider its direct limit X. It is easy to check that $lim_{k\to\infty}\Delta(X'_k, X) = 0$. Since $(X_n)_n$ is Cauchy this implies $lim_{k\to\infty}\Delta(X_k, X) = 0$. □

From Proposition 4.5 and Theorem 4.1 the following is immediate.

Theorem 4.6 *Let $T : \mathcal{C} \to \mathcal{C}$ be a contraction. Then there exists a "weak" fixed point of T, i.e. a complete metric space X (not necessarily unique) such that $\Delta(X, TX) = 0$.* □

Not every contraction $T : \mathcal{C} \to \mathcal{C}$ has a 'real' fixed point X in the sense that $TX \cong X$. For example consider T defined as follows: $TX = \tilde{X}$ for each $X \neq \tilde{X}$; $T\tilde{X} = \tilde{Y}$, where $\tilde{X}$ and $\tilde{Y}$ are defined as in remark 4.4. T is a contraction since for every X and Y, $\Delta(TX, TY) = 0$, but it does not have a fixed point X such that $TX \cong X$.

We now turn our attention to the compact case. Let Δ here denote the restriction of the pseudo-metric on $\mathcal{C}$ to the class of compact metric spaces $\mathcal{K}$.

Proposition 4.7 *The class $\mathcal{K}$ with distance Δ is a pseudo-metric space in which, for compact metric spaces X and Y,*

$$\text{if } \Delta(X, Y) = 0 \text{ then } X \cong Y.$$

Proof. Consider X and Y with $\Delta(X, Y) = 0$. Then there exists a sequence $(\langle f_n, g_n \rangle)_n$ of pairs of functions between them such that $\lim \delta(\langle f_n, g_n \rangle) = 0$. Because X and Y are compact there exists a (componentwise) converging subsequence, the limit of which is a proper adjoint pair, and hence an isomorphism. □

As for $\mathcal{C}$, we have the following.

Proposition 4.8 *The class $\mathcal{K}$ with distance Δ is complete (in the usual sense).*

The proof is as before, with the additional observation that the direct limit of a Cauchy tower of compact spaces is again compact. (See [4].)

By Theorem 4.1, Proposition 4.7, and Proposition 4.8, the following is now immediate.

Theorem 4.9 *Let $T : \mathcal{K} \to \mathcal{K}$ be a contraction. Then there is a compact metric space X such that $TX \cong X$. Moreover the fixed point is unique up to isomorphism.* □

Thus unique fixed points of contractions—which need not be functorial—on $\mathcal{K}$ are obtained, by applying Banach's fixed point theorem (for pseudo-metric spaces). As was announced in Section 3, Theorem 3.12 is a corollary of the theorem above, because contractive functors on $\mathbf{KMS}^{\approx}$ are contractions on $\mathcal{K}$.

Acknowledgement

The authors wish to thank Furio Honsell for discussions and suggestions; and Franck van Breugel, Marco Forti, and two anonymous referees for their comments.

References

[1] F. Alessi, P. Baldan, and G. Bellè. A fixed point theorem in a category of compact metric spaces. To appear in TCS.

[2] A. Arnold and M. Nivat. Metric interpretations of infinite trees and semantics of nondeterministic recursive programs. *Theoretical Computer Science*, 11(2):181–205, 1980.

[3] P. America and J. J. M. M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer and System Sciences*, 39:343–375, 1989.

[4] F. van Breugel and J. H. A. Warmerdam. Solving domain equations in a category of compact metric spaces. Report CS-R9424, CWI, Amsterdam, April 1994.

[5] J. W. de Bakker and J. I. Zucker. Processes and the denotational semantics of concurrency. *Information and Control*, 54(1/2):70–120, July/August 1982.

[6] B. Flagg and R. Kopperman. Continuity spaces: reconciling domains and metric spaces. To appear, 1995.

[7] M. Forti, F. Honsell, M. Lenisa: *Processes and Hyperuniverses.* Proceedings of the 19th Symposium on Mathematical Foundations of Computer Science 1994, volume 841 of LNCS, pages 352-361, Kŏsice, Slovakia, 1994, Springer-Verlag.

[8] R. E. Kent. The metric closure powerspace construction. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the 3rd Workshop on Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lecture Notes in Computer Science*, pages 173–199, New Orleans, April 1987. Springer-Verlag.

[9] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135–166, 1973.

[10] M. E. Majster-Cederbaum. The contraction property is sufficient to guarantee the uniqueness of fixed points in a category of complete metric spaces. *Information Processing Letters*, 33(1):15–17, 1989.

[11] J. J. M. M. Rutten and D. Turi. On the foundations of final semantics: non-standard sets, metric spaces, partial orders. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Proceedings of the REX Workshop on Semantics: Foundations and Applications*, volume 666 of *Lecture Notes in Computer Science*, pages 477–530, Beekbergen, June 1992. Springer-Verlag. FTP-available at `ftp.cwi.nl` as pub/CWIreports/AP/CS-R9241.ps.Z.

[12] J. J. M. M. Rutten. Elements of generalized ultrametric domain theory. Report CS-R9507, CWI, Amsterdam, 1995. FTP-available at `ftp.cwi.nl` as pub/CWIreports/AP/CS-R9507.ps.Z.

[13] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal of Computation*, 11(4):761–783, 1982.

[14] K. R. Wagner. *Solving recursive domain equations with enriched categories.* PhD thesis, Carnegie Mellon University, Pittsburgh, July 1994. Technical report CMU-CS-94-159.